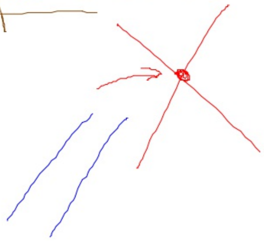
Chapter 7: Systems of Linear equations

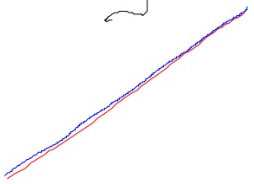
Summary:

system of 2 linear equations with 2 variables → one unique sol.

→ no solution

→ infinitely many solutions





Exercises: Solve & Graph the following linear systems:

c) $\boxed{x+y=0}$

$\boxed{2x+2y=-4}$

Substitution method:

Since $x+y=0$

$x=-y$

$2x+2y=-4$

$2(-y)+2y=-4$

$-2y+2y=-4$

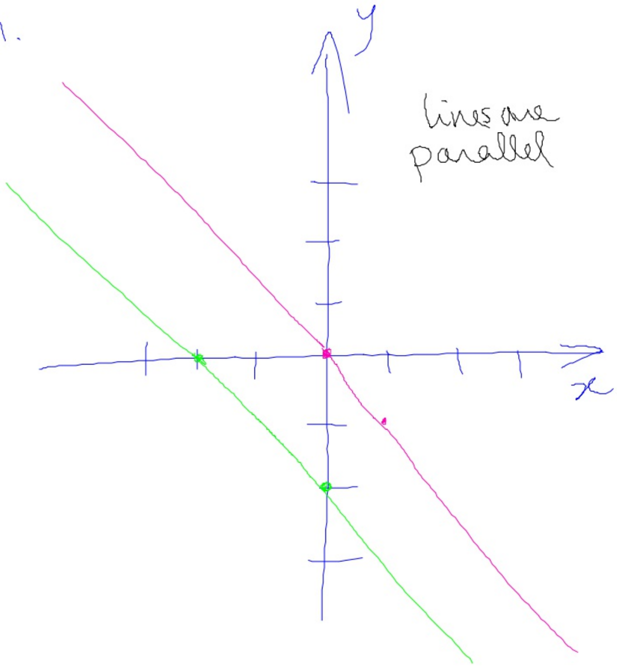
$0=-4$

No solution.

Graph:

x	0	1
y	0	-1

x	0	-2
y	-2	0



lines are parallel

d) $\left\{\begin{array}{l} \boxed{2x+4y=2} \\ \boxed{x+y=1} \end{array}\right.$ (-2)

Elimination method:

$$\begin{array}{r} 2x+4y=2 \\ -2x-2y=-2 \\ \hline 2y=0 \\ \boxed{y=0} \end{array}$$

Replacing $y=0$ in ~~each~~ any of the equations: $x+y=1$

$x+0=1$

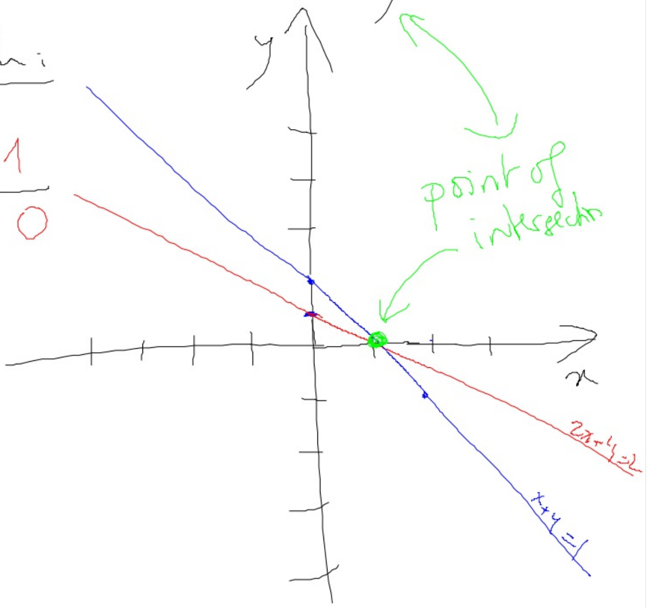
$\boxed{x=1}$

One unique solution $(1;0)$

Graph:

x	0	1
y	$1/2$	0

x	0	2
y	1	-1



e) $(-2)\begin{cases} x+y=2 \\ 2x+2y=4 \end{cases}$

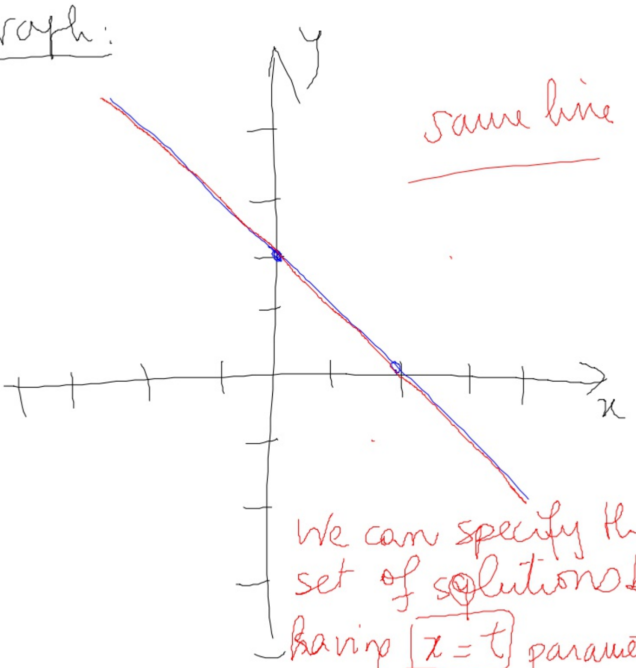
Elimination method:

$$\begin{array}{r} -2x-2y=-4 \\ 2x+2y=4 \\ \hline 0=0 \end{array}$$

Infinitely many sol.

(~~for all x~~ for any value of x,
I will a value for y that satisfy
both equations)

Graph:



same line

We can specify the set of solutions by:

having $\boxed{x=t}$ parameter

$y=2-x$

$\boxed{y=2-t}$

Solution: $\{ (t ; 2-t) ; t$ parameter real number $\}$

Linear Systems in n-variables

Solution by Row Operations:

R1 $4x - y + 2z = 13$

R2 $x + 2y - 2z = 0$

R3 start $\boxed{-x} + y + z = 5$

$$4x - y + 2z = 13$$
$$x + 2y - 2z = 0$$

$1R_2 + R_3$ $0 + 3y - z = +5$

Row operation Type I

* interchange ~~row~~ 2 rows

Row operation Type II

* multiply any row by a scalar

Row operation Type III

* multiply a row by a scalar and you add to another row, that replaces that last one.

$R_1 \quad 4x - y + 2z = 13$

$R_2 \quad \boxed{x} + 2y - 2z = 0$

$R_3 \quad 3y - z = 5$

$4x - y + 2z = 13$

$-4R_2 \quad \boxed{-4x} - 8y + 8z = 0$

$3y - z = 5$

$4x - y + 2z = 13$

$R_1 + R_2 \quad \boxed{\begin{array}{l} -9y + 10z = 13 \\ 3y - z = 5 \end{array}}$

$4x - y + 2z = 13$

$3R_3 + R_2 \quad 7z = 28$

$3y - z = 5$

Now, I can start finding x, y, z:

$7z = 28 \Rightarrow \boxed{z = 4}$

replacing z in the 3rd equation:

$3y - z = 5$

$3y - 4 = 5$

$3y = 9$

$\boxed{y = 3}$

Replacing y & z in the 1st equation:

$4x - 3 + 8 = 13$

$4x = 8$

$\boxed{x = 2}$

Solution: $(2, 3, 4)$

Ex:

$-x - y + z = -2$

$3x + 2y - 2z = 7$

$\boxed{x} + 3y - 3z = 0$

$-x - y + z = -2$

$\boxed{3x} + 2y - 2z = 7$

$R_1 + R_3:$ $2y - 2z = -2$

$-x - y + z = -2$

$3R_1 + R_2:$ $-y + z = 1$

$\boxed{2y} - 2z = -2$

$-x - y + z = -2$

$-y + z = 1$

$2R_2 + R_3:$ $0 = 0$

$-x - y + z = -2$

$-y + z = 1$

$-x - y + z = -2$

$-R_1 + R_2:$ $x + 0 + 0 = 3$

$x = 3$

$-3 - y + z = -2$

$-y + z = 1$

$$-y + z = 1$$
$$-y + z = 1$$

$$-y + z = 1$$

$-R_1 + R_2$: $0 + 0 = 0$

$$-y + z = 1$$

I set $\boxed{z = t}$ parameter

$$y = z - 1$$
$$\boxed{y = t - 1}$$
$$\boxed{x = 3}$$

Solution:

$(3, t-1, t)$; t parameter

Ex:

$$\begin{aligned} -x - y - z &= 0 \\ x + y + z &= 7 \\ 2x - 3y &= 0 \end{aligned}$$

$$\begin{aligned} -x - y - z &= 0 \\ R_1+R_2 \quad 0 + 0 + 0 &= 7 \\ 2x - 3y &= 0 \end{aligned}$$

$0 = 7$ is not possible

Therefore the system has no solution,

the system is <u>inconsistent</u>

Remark: A linear system has

- → no solution
- → exactly one unique solution
- → infinitly many solutions.

Matrices:

$$\begin{cases} 4x - y + 2z = 13 \\ x + 2y - 2z = 0 \\ -x + y + z = 5 \end{cases} \iff \left[\begin{array}{c|c} \underbrace{\begin{matrix} x & y & z \\ 4 & -1 & 2 \\ 1 & 2 & -2 \\ \boxed{-1} & 1 & 1 \end{matrix}}_{\text{coefficient matrix}} & \begin{matrix} \\ 13 \\ 0 \\ 5 \end{matrix} \end{array}\right]$$

$$R_2+R_3 \left[\begin{array}{ccc|c} 4 & -1 & 2 & 13 \\ \boxed{1} & 2 & -2 & 0 \\ 0 & 3 & -1 & 5 \end{array}\right] \quad \frac{-1}{4}R_1+R_2 \left[\begin{array}{ccc|c} 4 & -1 & 2 & 13 \\ 0 & 9/4 & -5/2 & -13/4 \\ 0 & 3 & -1 & 5 \end{array}\right]$$

$$4R_2 \left[\begin{array}{ccc|c} 4 & -1 & 2 & 13 \\ 0 & 9 & -10 & -13 \\ 0 & \boxed{3} & -1 & 5 \end{array}\right] \quad -3R_3 \left[\begin{array}{ccc|c} 4 & -1 & 2 & 13 \\ 0 & 9 & -10 & -13 \\ 0 & \boxed{-9} & 3 & -15 \end{array}\right] \quad R_2+R_3 \left[\begin{array}{ccc|c} 4 & -1 & 2 & 13 \\ 0 & 9 & -10 & -13 \\ 0 & 0 & -7 & -28 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 4 & -1 & +2 & 13 \\ 0 & 9 & -10 & -13 \\ 0 & 0 & -7 & -28 \end{array}\right]$$

①

I write it again as a linear system.

$$\begin{cases} 4x - y + 2z = 13 \\ 9y - 10z = -13 \\ -7z = -28 \end{cases}$$

so $z = 4$
$y = 3$
$x = 2$

one solution

②

$$-\frac{1}{7}R_3 \left[\begin{array}{ccc|c} 4 & -1 & 2 & 13 \\ 0 & 9 & \boxed{-10} & -13 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

diagonal

My plan here is to have all entries "zero", except the diagonal.

$$10R_3+R_2 \left[\begin{array}{ccc|c} 4 & \boxed{-1} & 2 & 13 \\ 0 & 9 & 0 & 27 \\ 0 & 0 & 1 & 4 \end{array}\right] \quad \frac{1}{9}R_2+R_1 \left[\begin{array}{ccc|c} 4 & 0 & \boxed{2} & 16 \\ 0 & 9 & 0 & 27 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 4 & 0 & \boxed{2} & 16 \\ 0 & 9 & 0 & 27 \\ 0 & 0 & 1 & 4 \end{array}\right] \qquad -2R_3+R_1 \left[\begin{array}{ccc|c} 4 & 0 & 0 & 8 \\ 0 & 9 & 0 & 27 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Making all diagonal entries equal to 1 :

$$\begin{array}{l} \frac{1}{4}R_1 \\ \frac{1}{9}R_2 \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right] \begin{array}{l} \Rightarrow x=2 \\ \Rightarrow y=3 \\ \Rightarrow z=4 \end{array}$$

One solution $(2;3;4)$

$\underline{\text{Ex}}$. Solve the following equations by row operations, using matrices:

$$\begin{cases} 2x + z = 10 \\ 2y = z \\ z - 3y = 6x \end{cases} \qquad \begin{cases} 2x + z = 10 \\ 2y - z = 0 \\ -6x - 3y + z = 0 \end{cases} \qquad \left(\begin{array}{ccc|c} 2 & 0 & 1 & 10 \\ 0 & 2 & -1 & 0 \\ \boxed{-6} & -3 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 0 & 1 & 10 \\ 0 & 2 & -1 & 0 \\ 0 & \boxed{-3} & 4 & 30 \end{array}\right) 3R_1 + R_3 \qquad \frac{3}{2}R_2 + R_3 \left(\begin{array}{ccc|c} 2 & 0 & \boxed{1} & 10 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & \frac{5}{2} & 30 \end{array}\right) \qquad -\frac{2}{5}R_3 + R_1 \left(\begin{array}{ccc|c} 2 & 0 & 0 & -2 \\ 0 & 2 & \boxed{-1} & 0 \\ 0 & 0 & 5/2 & 30 \end{array}\right)$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & -2 \\ 0 & 2 & \boxed{-1} & 0 \\ 0 & 0 & 5/2 & 30 \end{array}\right] \quad \frac{2}{5}R_3 + R_2 \quad \left[\begin{array}{ccc|c} 2 & 0 & 0 & -2 \\ 0 & 2 & 0 & 12 \\ 0 & 0 & 5/2 & 30 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 12 \end{array}\right]$$

$x = -1$

$y = 6 \Rightarrow$ one solution

$z = 12$

$(-1, 6, 12)$

Exercises:

$$\begin{cases} 4x - 2y + 2z = 6 \\ -y + z = 1 \\ 2x = 2 \end{cases} \qquad \left[\begin{array}{ccc|c} 4 & -2 & 2 & 6 \\ 0 & -1 & 1 & 1 \\ 2 & 0 & 0 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & -1 & 1 & 1 \\ \boxed{4} & -2 & 2 & 6 \end{array}\right] \xrightarrow{-2R_1+R_3} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & -1 & 1 & 1 \\ 0 & \boxed{-2} & 2 & 2 \end{array}\right] \xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} 2x = 2 \\ -y + z = 1 \end{cases} \qquad \begin{cases} x = 1 \\ -y + z = 1 \end{cases}$$

$x = 1$

let $z = t$ parameter

$y = t - 1$

solution is the set:

$\{(1; t-1; t);$ t parameter$\}$

HW:

$$\begin{cases} -x_1 + 2x_2 - 3x_3 = 2 \\ \quad\; -2x_2 \qquad\quad = 3 \\ 2x_1 - x_2 + x_3 = 9 \end{cases}$$

$$\underbrace{\left[\begin{array}{c|c} \underbrace{\begin{matrix} -1 & 2 & -3 \\ 0 & -2 & 0 \\ \boxed{2} & -1 & 1 \end{matrix}}_{\text{coef matrix}} & \begin{matrix} 2 \\ 3 \\ 9 \end{matrix} \end{array}\right]}_{\text{augmented matrix}}$$

$$2R_1 + R_3 \left[\begin{array}{ccc|c} -1 & 2 & -3 & 2 \\ 0 & -2 & 0 & 3 \\ 0 & \boxed{3} & -5 & 13 \end{array}\right]$$

$$\frac{3}{2}R_2 + R_3 \left[\begin{array}{ccc|c} -1 & \boxed{2} & -3 & 2 \\ 0 & -2 & 0 & 3 \\ 0 & 0 & -5 & \frac{35}{2} \end{array}\right]$$

$$\frac{9}{2} + 13 = \frac{35}{2}$$

$$\left[\begin{array}{ccc|c} -1 & \boxed{2} & -3 & 2 \\ 0 & -2 & 0 & 3 \\ 0 & 0 & -5 & \frac{35}{2} \end{array}\right]$$

$$R_2+R_1 \left[\begin{array}{ccc|c} -1 & 0 & \boxed{-3} & 5 \\ 0 & -2 & 0 & 3 \\ 0 & 0 & -5 & \frac{35}{2} \end{array}\right] \begin{array}{c} \leftarrow \\ \\ \leftarrow \end{array}$$

$$-\frac{3}{5}R_3+R_1 \left[\begin{array}{ccc|c} -1 & 0 & 0 & -\frac{11}{2} \\ 0 & -2 & 0 & 3 \\ 0 & 0 & -5 & \frac{35}{2} \end{array}\right]$$

$$\begin{array}{c} -R_1 \\ -\frac{1}{2}R_2 \\ -\frac{1}{5}R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{11}{2} \\ 0 & 1 & 0 & -\frac{3}{2} \\ 0 & 0 & 1 & -\frac{7}{2} \end{array}\right]$$

$x_1 = \frac{11}{2}$

$x_2 = -\frac{3}{2}$

$x_3 = -\frac{7}{2}$

Solution: $\left(\frac{11}{2}, -\frac{3}{2}, -\frac{7}{2}\right)$

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 1 \\ -x_1 + x_2 \qquad - x_4 = 1 \\ x_1 - x_2 + x_3 \qquad = 1 \\ \qquad x_2 + x_3 \qquad = 1 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & -1 & 1 \\ 1 & -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$R_2+R_3 \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$R_2 \leftrightarrow R_4 \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ -1 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$-R_4+R_1$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ -1 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$2R_4 \left[\begin{array}{cccc|c} 2 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ -2 & 2 & 0 & -2 & 2 \end{array}\right]$$

$$R_1+R_4 \left[\begin{array}{cccc|c} 2 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & \boxed{2} & 1 & 0 & 2 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & -1 & 0 & 0 \end{array}\right] -2R_2+R_4$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 2 \\ 0 & 0 & \boxed{-1} & 0 & 0 \end{array}\right]$$

$$-R_3+R_4 \left[\begin{array}{cccc|c} 2 & 0 & \boxed{1} & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & -1 & -2 \end{array}\right]$$

$$R_3+R_1 \left[\begin{array}{cccc|c} 2 & 0 & 0 & \boxed{3} & 2 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & -1 & -2 \end{array}\right]$$

$$3R_4+R_1 \left[\begin{array}{cccc|c} 2 & 0 & 0 & 0 & -4 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & -1 & -2 \end{array}\right]$$

$$R_4+R_3 \left[\begin{array}{cccc|c} 2 & 0 & 0 & 0 & -4 \\ 0 & 1 & \boxed{1} & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -2 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -2 \end{array}\right] R_3+R_2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{cases} x_1 = -2 \\ x_2 = 1 \\ x_3 = 0 \\ x_4 = 2 \end{cases}$$

$$\begin{cases} x \boxed{+ y + z} = 0 \\ 2x + 2y - z = 0 \\ \boxed{y + z} = 0 \end{cases}$$

$x + 0 - 0 \Rightarrow x - 0$

$$\begin{array}{r} 2y - z = 0 \\ + \quad y + z = 0 \\ \hline 3y = 0 \Rightarrow y = 0 \\ z = 0 \end{array}$$

sol

$(0; 0, 0)$